NASA TN D-1563

N63-12195
Code 1

NASA TN D-1563

TECHNICAL NOTE

D-1563

FEASIBILITY OF APPLYING FIELD-ION EMISSION TO ELECTROSTATIC ROCKET ENGINES

By N. Stankiewicz

Lewis Research Center
Cleveland, Ohio

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

WASHINGTON January 1963

Copy

Code 1

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

TECHNICAL NOTE D-1563

FEASIBILITY OF APPLYING FIELD-ION EMISSION TO ELECTROSTATIC ROCKET ENGINES

By N. Stankiewicz

SUMMARY

The Wentzel, Kramers, Brillouin method was used to determine the ionization lifetimes of hydrogen, lithium, sodium, rubidium, cesium, and xenon in an applied electric field. These lifetimes were used in the analysis of a theoretical plane-diode engine consisting of a grid emitter that field-ionizes the impinging propellant. Results show that monatomic gaseous propellants are not suitable for field-ion emission engines of this design because of the excessively high voltage requirements for ionization.

INTRODUCTION

The power expended in the ionization of a propellant is a primary source of inefficiency in electrostatic rocket engines that employ contact ionization or electron bombardment. An important improvement in engine performance could be effected if ionization could take place at ambient temperatures and nearly 100 percent propellant utilization efficiency. The idea of field-emission ionization arises naturally from these considerations. It is the purpose of this paper to examine the feasibility of applying this effect to electrostatic rocket engines.

The lifetimes of hydrogen, lithium, sodium, rubidium, cesium, and xenon in a high electric field are calculated by using the Wentzel, Kramers, Brillouin (W.K.B.) approximation. Image potentials of the ion and electron caused by the anode surface were neglected. Omission of these image forces can be justified if ionization takes place relatively far from the surface. A coulomb attraction between the outer electron and the remaining atom was assumed for all the elements; this is correct except for xenon. The resulting error for xenon is minor since the prominent terms in the W.K.B. approximation are the field strength and the ionization potential.

The Bohr theory is used to calculate the frequency of the electron in its orbit, that is, the number of collisions per second that the electron makes with the potential barrier created by the superposition of the coulomb field and the external electric field.

The effective capture diameter of an infinitely long, charged, cylindrical wire is computed for a collimated beam of neutral atoms of given polarizability. The ionization probability is calculated for these induced dipoles, whose force of attraction is into the high field region, and yields the relation between field strength and cylinder radius necessary for ionization.

A theoretical engine is analyzed in which the emitting surface is a plane grating of fine wires. It is shown that the space-charge-limited current is equivalent to that of a plane diode in which the grid acts as the emitting plane.

THEORETICAL BACKGROUND

Two types of phenomena are associated with the introduction of an atom into an electric field. First, the degeneracy of the energy levels (with the same principal quantum number n but with different orbital quantum numbers l) can be removed, giving rise to the Stark effect. Second, if the fields are high enough, the phenomenon of autoionization, or electron tunneling through the potential barrier, will take place within a finite length of time. The quantum mechanical formulation of these related effects will now be very briefly reviewed.

The Schrödinger equation for a hydrogen-like atom (neglecting spin-orbit interaction) in a constant external field $\mathscr{F}$ is

$$-\frac{\hbar^2}{2\mu}\nabla^2\psi + \left(-\frac{Ze^2}{4\pi\epsilon_0 r} - e\mathscr{F}z\right)\psi = E\psi \tag{1}$$

(Symbols are defined in appendix A, and physical constants are presented in table I.) Written in parabolic coordinates,

$$\left.\begin{aligned} x &= \xi\eta\cos\varphi \\ y &= \xi\eta\sin\varphi \\ z &= \frac{1}{2}(\eta^2 - \xi^2) \end{aligned}\right\} \tag{2}$$

equation (1) separates into:

$$\frac{d^2\Phi}{d\varphi^2} + K_1^2\Phi = 0 \tag{3a}$$

$$\frac{1}{\xi}\frac{d}{d\xi}\left(\xi\frac{dL}{d\xi}\right) + \frac{2\mu}{\hbar^2}\left(-\frac{e\mathscr{F}}{2}\xi^4 + E\xi^2 - \frac{\hbar^2K_1^2}{2\mu}\frac{1}{\xi^2} - K_2\right)L = 0 \tag{3b}$$

$$\frac{1}{\eta}\frac{d}{d\eta}\left(\eta\frac{dH}{d\eta}\right) + \frac{2\mu}{\hbar^2}\left(\frac{e\mathscr{F}}{2}\eta^4 + E\eta^2 - \frac{\hbar^2K_1^2}{2\mu}\frac{1}{\eta^2} - K_3\right)H = 0 \tag{3c}$$

$$K_2 + K_3 + \left(\frac{Ze^2}{2\pi\epsilon_0}\right) = 0 \tag{3d}$$

Here $\psi(\xi,\eta,\varphi) = L(\xi)H(\eta)\Phi(\varphi)$, and K_1, K_2, K_3 are separation constants.

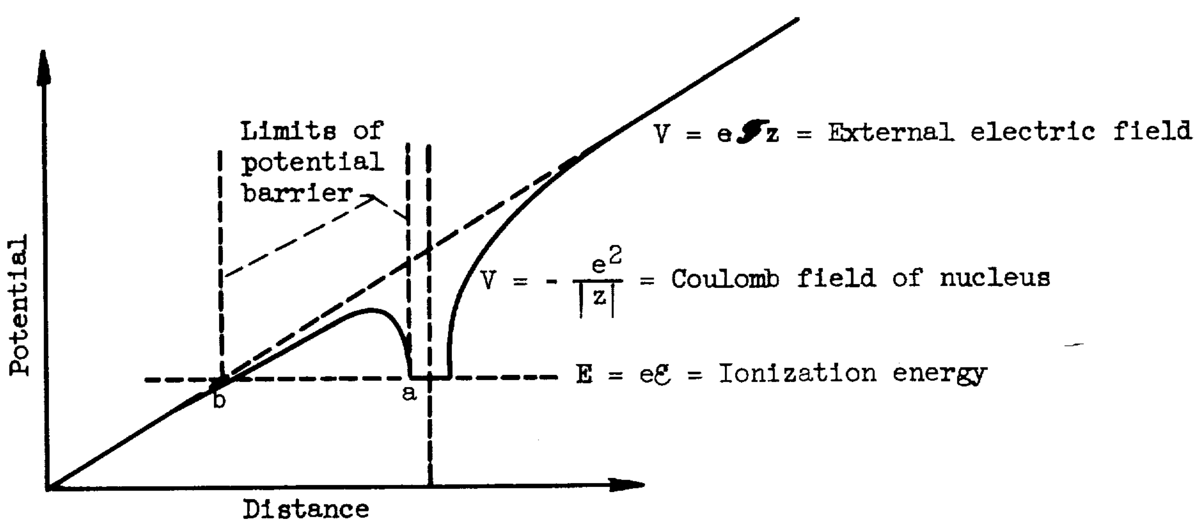
Notice that equation (3b) contains terms that go to $-\infty$ for infinite values of the coordinate ξ. Reference 1 shows that such an equation gives no quadratically integrable solutions and contains no stable stationary states; that is, the effect of the electric field is to give aperiodic wave functions, and hence a definite probability of finding the electron at large distances from the nucleus. This seems contrary to what is observed in the Stark effect, namely, definite spectral lines indicating periodic wave functions that give a vanishing probability of finding the electron at infinity. In reference 2, however, the lifetime of this unstable state is calculated for the case of hydrogen and has a value of $10^{10^{10}}$ seconds for a field of 1 volt per centimeter. The lifetimes are still extremely long for the fields encountered in the Stark effect, and an observer need never be aware of the autoionization phenomena. Spectral lines due to transitions with higher principal quantum numbers have been seen to disappear under the influence of an electric field.

EQUIVALENT ONE-DIMENSIONAL CASE

Because of the obvious difficulty in finding exact solutions to equation (1), simplifying assumptions are invariably made. A good approximation (see refs. 3 and 4) of the ionization time is obtained by solving the one-dimensional problem, namely

$$-\frac{\hbar^2}{2m_e}\frac{d^2\psi}{dz^2} + \left(-\frac{e^2}{4\pi\epsilon_0|z|} - e\mathcal{F}z\right)\psi = E\psi = -e\mathcal{E}\psi \qquad (4)$$

and applying the W.K.B. approximation (see ref. 5). The situation is depicted in the following potential diagram, which shows the external field superimposed on the coulomb field:



(a)

The penetration through the potential barrier is then given by

$$D = \exp\left[-\frac{2\sqrt{2m_e}}{\hbar}\int_a^b (V - E)^{1/2}\,dz\right]$$

$$= \exp\left[-\frac{2\sqrt{2m_e}}{\hbar}\int_a^b \left(e\mathcal{E} - \frac{e^2}{4\pi\epsilon_0|z|} - e\mathscr{F}z\right)^{1/2} dz\right] \tag{5}$$

The lifetime is found by taking the reciprocal of the penetration probability D multiplied by the frequency ν of the electron striking the barrier, that is

$$\tau = \frac{1}{\nu D} \tag{6}$$

The limits on the integral give the end points of the barrier as shown in sketch (a). Appendix B gives the solution of the integral and the ionization time, in mks units, as

$$\tau = \nu^{-1}\exp\left\{\frac{4\sqrt{m_e e}}{3\hbar}\frac{\mathcal{E}^{3/2}}{\mathscr{F}}C_+^{1/2}\left[E\left(\frac{\pi}{2},k\right) - C_-K\left(\frac{\pi}{2},k\right)\right]\right\} \tag{7}$$

where

$$C_\pm = 1 \pm \left(1 - \frac{e\mathscr{F}}{\pi\epsilon_0\mathcal{E}^2}\right)^{1/2}$$

$$k = \left(\frac{C_+ - C_-}{C_+}\right)^{1/2}$$

$K\left(\frac{\pi}{2},k\right)$ is the complete elliptic integral of the first kind, $E\left(\frac{\pi}{2},k\right)$ is the complete elliptic integral of the second kind, and $\mathcal{E}$ is the ionization potential in volts.

A maximum field strength can be defined by letting $\mathscr{F}_{max} = \frac{\pi\epsilon_0\mathcal{E}^2}{e}$ It is at this field strength that the barrier disappears and $\tau = \nu^{-1}$.

The orbital frequency ν is obtained by recourse to Bohr's theory, which gives for hydrogen and the alkali metals

$$\nu = \frac{1}{n^3}\frac{m_e e^4}{2\pi\hbar^3(4\pi\epsilon_0)^2} \tag{8}$$

Table I gives the values of the principal quantum number n and the ionization potential $\mathcal{E}$ for the elements considered. Figure 1 shows the logarithm of the lifetimes plotted against field strength of these elements.

TABLE I. - FUNDAMENTAL AND DERIVED CONSTANTS

(a) Miscellaneous properties of elements discussed

Element	Chemical symbol	Principal quantum number n	Ionization potential, $\mathscr{E}$, v	Atomic mass, m, kg	Polarizability, α, (f)(m²) (a)	Boiling point (at 760 mm Hg), T_{bp}, °K	Field energy of dipole at T_{bp} and $\mathscr{F}$ = 1 v/Å, ϵ_1
Hydrogen[b]	H	1	13.595	1.673×10^{-27}	---------	----	-----
Lithium	Li	2	5.390	1.151×10^{-26}	1.3×10^{-39}	1609	6.01
Sodium	Na	3	5.138	3.82×10^{-26}	3.0×10^{-39}	1153	18.84
Rubidium	Rb	5	4.176	1.419×10^{-25}	5.6×10^{-39}	973	41.4
Cesium	Cs	6	3.893	2.205×10^{-25}	4.6×10^{-39}	943	35.7
Xenon[c]	Xe	-	12.127	2.18×10^{-25}	4.6×10^{-40}	166	11.09

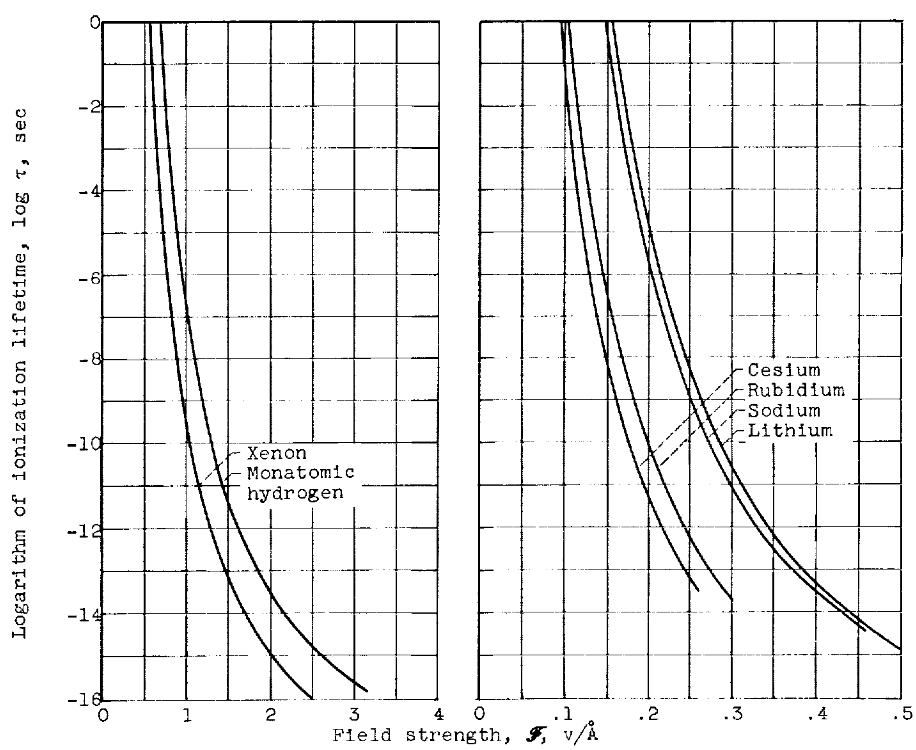
(b) Physical constants

Avogadro's number, N_0, atoms/kg atomic wt	6.0251×10^{26}
Electronic charge, e, coulombs	1.602×10^{-19}
Electronic mass, m_e, kg	9.11×10^{-31}
Planck's constant, h, (j)(sec)	6.6237×10^{-34}
Boltzmann's constant, k, j/°K	1.38×10^{-23}
Vacuum dielectric constant, ϵ_0, f/m	$10^7/4\pi c^2$

[a]Ref. 8.

[b]Monatomic hydrogen.

[c]Orbital frequency assumed to be 10^{16} sec^{-1} for xenon.



(a) Xenon and monatomic hydrogen.

(b) Cesium, rubidium, sodium, and lithium.

Figure 1. - Effect of electric-field strength on ionization lifetimes according to W.K.B. approximation.

SINGLE-WIRE IONIZER

There are two problems connected with the construction of an efficient ionizer using field-ion emission. First, a very high field must be established by some combination of electrode geometry and applied voltage. Since there are usually other restrictions that limit the voltage, the high field must then be produced by employing electrodes of minute proportions. This is somewhat unfortunate because the strong fields now exist only in the rather restricted region of the immediate vicinity of the electrode. This presents the second problem. The incoming gas must be made to remain in this "ionization zone" for a sufficient time to allow ionization to take place. The situation is remedied somewhat because of the dipole moment induced in the particles by the field and their resultant attraction into the strong field region. Since this energy of attraction is proportional to the square of the field strength, it is obvious that a point electrode would produce a shorter range force ($\sim 1/r^5$) than a wire electrode ($\sim 1/r^3$) and that the former would give a much smaller capture cross section for ionization. For this reason this section considers only the wire emitter.

The electric field surrounding an infinitely long cylindrical wire of radius r_a can be written

$$\mathscr{F} = \frac{\mathscr{F}_a r_a}{r} \tag{9}$$

where $\mathscr{F}_a$ is the field strength at the wire surface.

The potential energy U of an induced dipole in such a field is

$$U = \frac{-\alpha \mathscr{F}^2}{2} = -\alpha \frac{\mathscr{F}_a^2 r_a^2}{2r^2} \tag{10}$$

Here α is the polarizability; values are given in table I. In general, the differential equation defining the orbit of a particle in a central field is given by

$$d\theta = \frac{dr}{r^2\left(\frac{2m_pE}{l^2} - \frac{2m_pU}{l^2} - \frac{1}{r^2}\right)^{1/2}} \tag{11}$$

where l is the angular momentum of the particle; E, the particle energy, is equal to its thermal energy of translation and is related to the angular momentum by

$$l = s\sqrt{2m_pE} \tag{12}$$

where s is the impact parameter.

Substituting equations (10) and (12) into (11) results in

$$d\theta = \frac{dr}{r^2\left[\frac{1}{s^2} - \left(\frac{s^2E - \frac{1}{2}\alpha\mathcal{F}_a^2 r_a^2}{s^2E}\right)\frac{1}{r^2}\right]^{1/2}} \tag{13}$$

Integration then yields the equation of the orbit:

$$r = \left(\frac{s^2E - \frac{1}{2}\alpha\mathcal{F}_a^2 r_a^2}{E}\right)^{1/2} \frac{1}{\sin\left(\frac{s^2E - \frac{1}{2}\alpha\mathcal{F}_a^2 r_a^2}{s^2E}\right)^{1/2}\theta} \tag{14}$$

The minimum value of r occurs at

$$r_{min} = \left(\frac{s^2E - \frac{1}{2}\alpha\mathcal{F}_a^2 r_a^2}{E}\right)^{1/2}$$

If $r_{min} \leq r_a$, then a collision with the wire occurs. The resulting relation between the maximum energy and the impact parameter for a collision is

$$E_{max} = \frac{\frac{1}{2}\alpha\mathcal{F}_a^2 r_a^2}{s^2 - r_a^2}; \qquad s \geq r_a \tag{15}$$

Assume that the wire is placed in a collimated, neutral particle beam of flux density J_N whose energy distribution is Maxwellian. The differential flux density dJ_N with an energy between E and $E + dE$ is

$$dJ_N = \frac{2J_N}{\sqrt{\pi}} \frac{1}{(kT)^{3/2}} E^{1/2} e^{-E/kT} dE \tag{16}$$

The number of particles per second per unit wire length having an impact parameter between s and $s + ds$ and with an energy lying between E and $E + dE$ is then $2dJ_N\,ds$, the factor 2 occurs because both the top and the bottom of the wire are being considered. The total number of collisions is then found by integrating equation (16) over E up to E_{max} and over $s > r_a$ and then adding to this the number of particles having an impact parameter less than r_a:

$$\dot{N} = 2J_N r_a + \frac{4J_N}{\pi^{1/2}(kT)^{3/2}} \int_{r_a}^{\infty} \int_0^{\frac{\alpha\mathcal{F}_a^2}{2} r_a^2 \Big/ \left(s^2 - r_a^2\right)} E^{1/2} e^{-E/kT} dE\, ds \tag{17}$$

Carrying out the integration over E leaves

$$\dot{N} = 2J_N\left\{r_a + \int_{r_a}^{\infty} \operatorname{erf}\left[\frac{\frac{1}{2}\alpha\mathcal{F}_a^2 r_a^2}{kT\left(s^2 - r_a^2\right)}\right]^{1/2} ds - \int_{r_a}^{\infty} \frac{2}{\sqrt{\pi}}\left[\frac{\frac{1}{2}\alpha\mathcal{F}_a^2 r_a^2}{kT\left(s^2 - r_a^2\right)}\right]^{1/2} e^{-\frac{\frac{1}{2}\alpha\mathcal{F}_a^2 r_a^2}{kT(s^2 - r_a^2)}} ds\right\} \tag{18}$$

With the substitutions

$$\epsilon = \frac{\frac{1}{2}\alpha\mathcal{F}_a^2}{kT} \qquad y = \frac{s}{r_a} \tag{19}$$

equation (18) becomes

$$\begin{aligned}\dot{N} &= 2J_N r_a\left\{1 + \int_1^{\infty} \operatorname{erf}\left[\left(\frac{\epsilon}{y^2 - 1}\right)^{1/2}\right] dy - \frac{2}{\sqrt{\pi}}\int_1^{\infty}\left(\frac{\epsilon}{y^2 - 1}\right)^{1/2} e^{-\frac{\epsilon}{y^2 - 1}} dy\right\} \\ &= 2J_N R_a\end{aligned} \tag{20}$$

where R_a is the effective capture radius. A plot of R_a/r_a against ϵ is shown in figure 2.

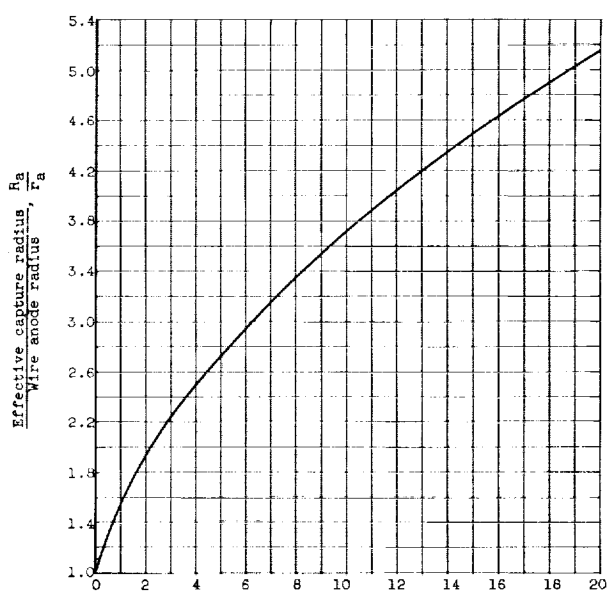


Figure 2. - Ratio of effective collision radius to wire anode radius as function of field energy of induced dipole for an infinitely long wire anode.

The probability that an atom will be ionized on passing through a region of high field strength is given by

$$P = 1 - e^{-\int \frac{dt}{\tau}} \tag{21}$$

A conservative estimate of the ionization efficiency of a beam of particles can be made by considering only the fast-moving ones in a direct collision with the wire. These particles will have the smallest residence time and hence will define a minimum ionization probability.

For an atom whose impact parameter is zero (no angular momentum) the velocity through the field is

$$-\frac{dr}{dt} = \sqrt{\frac{2}{m_e}\,(E - U)} \tag{22}$$

where U is the potential energy of an induced dipole given in equation (10) and E is the initial thermal energy of the atom. Therefore,

$$-\frac{dr}{dt} = \left[\frac{2}{m_e}\left(E + \frac{\alpha \mathscr{F}^2}{2}\right)\right]^{1/2} \tag{23}$$

The field surrounding the wire is given in equation (9), and differentiating gives the relation between dr and $d\mathscr{F}$:

$$dr = -\frac{\mathscr{F}_a r_a \, d\mathscr{F}}{\mathscr{F}^2} \tag{24}$$

In terms of $\mathscr{F}$ the transit time becomes

$$dt = \frac{\mathscr{F}_a r_a \, d\mathscr{F}}{\mathscr{F}^2\left[\frac{2}{m_e}\left(E + \frac{\alpha \mathscr{F}^2}{2}\right)\right]^{1/2}} \tag{25}$$

Substituting equation (25) into equation (21) gives the ionization probability entirely in terms of $\mathscr{F}$:

$$P = 1 - \exp\left\{-\mathscr{F}_a r_a \int_0^{\mathscr{F}_a} \frac{d\mathscr{F}}{\tau(\mathscr{F})\mathscr{F}^2\left[\frac{2}{m_e}\left(E + \frac{\alpha \mathscr{F}^2}{2}\right)\right]^{1/2}}\right\} \tag{26}$$

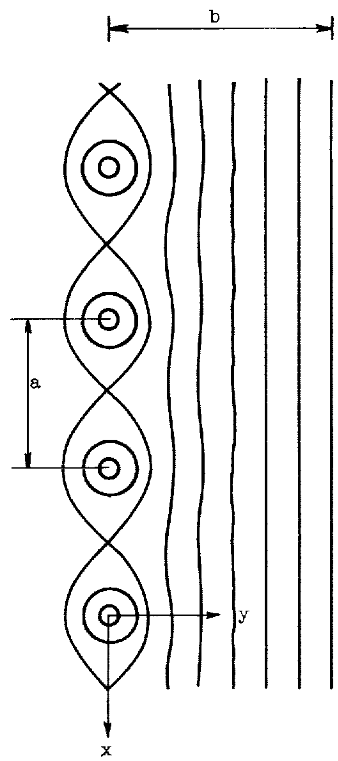
Setting the integral equal to unity gives an ionization probability of 63 percent. The wire radius necessary to achieve this probability is then determined by integrating over the field strength from zero to $\mathscr{F}_a$:

$$r_a = \left\{ \mathscr{F}_a \int_0^{\mathscr{F}_a} \frac{d\mathscr{F}}{\tau(\mathscr{F})\mathscr{F}^2\left[\frac{2}{m_e}\left(E + \frac{\alpha\mathscr{F}^2}{2}\right)\right]^{1/2}} \right\}^{-1} \tag{27}$$

The integral in equation (27) was computed numerically for $E = 4kT$ at the boiling point of the gas at atmospheric pressure; the results are given in figure 3.[1] Over 98 percent of the particles have energies less than $4kT$, according to the Boltzmann distribution, and of these at least 63 percent will be ionized in a direct collision if the relation in equation (27) is realized. For indirect collisions the particles remain in the field a longer time and, therefore, will also be ionized.

GRID IONIZER

The potential due to the uniform grid of wires (shown in sketch (b)) with a



(b)

[1]Hydrogen is not included in this calculation because its lifetime as plotted in figure 1(a) is for the monatomic gas rather than for its natural molecular state.

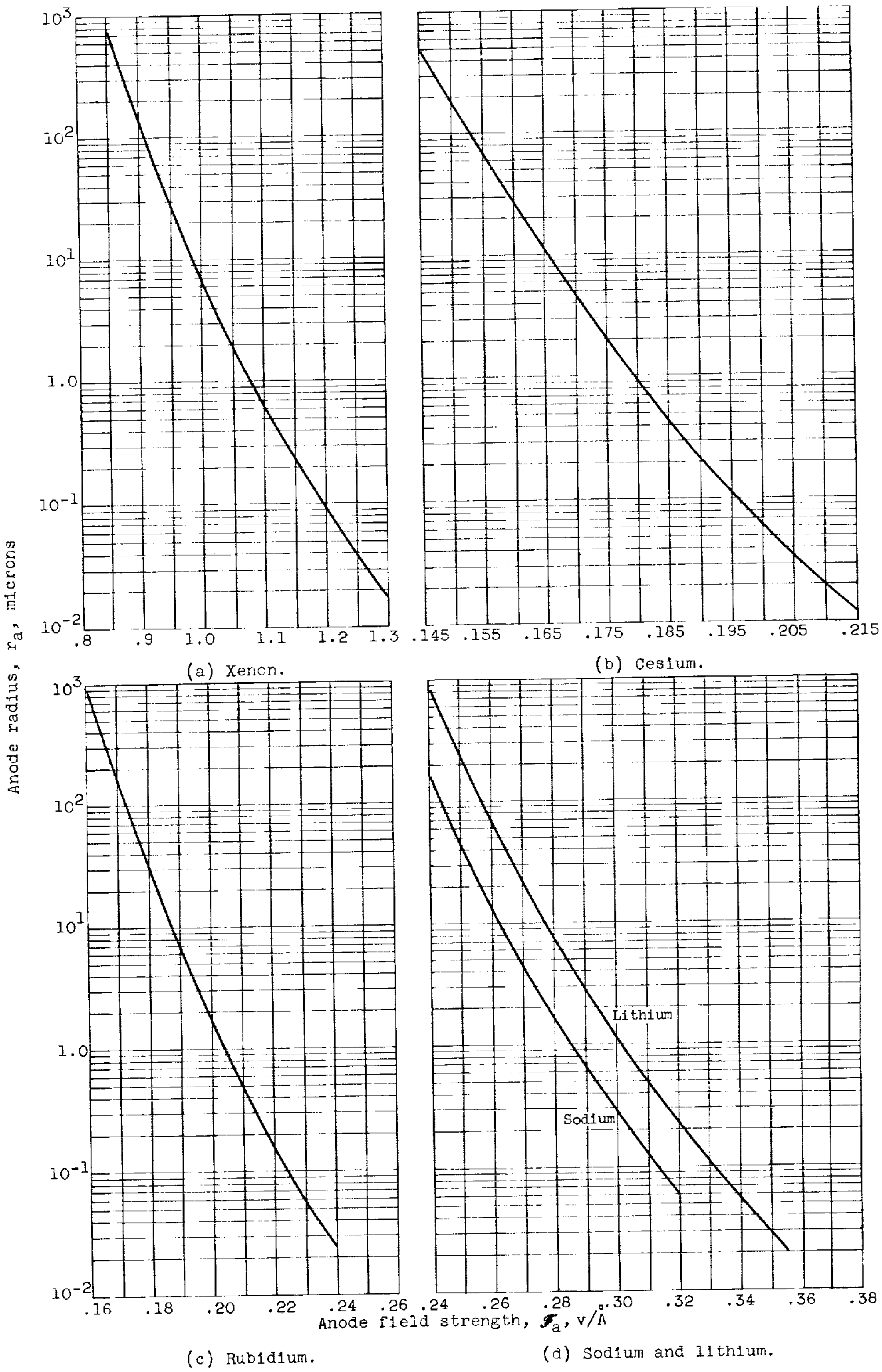


Figure 3. - Wire anode radius as function of anode field strength for various elements. Translational thermal energy of 4kT at atmospheric-pressure boiling temperature T, where k is the Boltzmann constant.

grid spacing a along the x-axis is given by the real part of the function (ref. 5)

$$W = -K \ln\left(2 \sin \frac{\pi z}{a}\right) \tag{28}$$

$$z = x + iy$$

The potential is, then,

$$V = V_a - K \ln 4\left(\sin^2 \frac{\pi}{a} x + \sinh^2 \frac{\pi}{a} y\right) \tag{29}$$

The field strength is also obtained from equation (28) and is

$$\mathscr{F} = -\frac{dW^*}{dz^*} = \frac{\pi}{a} K \cot \frac{\pi}{a} z^* \tag{30}$$

The asterisk denotes the complex conjugate. The constants K and V_a are evaluated in terms of the field strength on the surface of a grid wire and the arbitrary zero of potential. The field strength for small z is

$$\mathscr{F} \cong \frac{\pi}{a} K \frac{1}{\frac{\pi}{a} z^*} = \frac{K}{z^*} \tag{31}$$

At r_a the magnitude of the field strength is $\mathscr{F}_a$; thus, $K = \mathscr{F}_a r_a$. The zero of potential will be taken at a distance b from the grid, a distance equal to the accelerator spacing. Since $b \gg a$,

$$\sinh^2 \frac{\pi}{a} b \cong \frac{1}{4} e^{2\pi b/a} \gg \sin^2 \frac{\pi}{a} x$$

and therefore

$$V_a = \mathscr{F}_a r_a \frac{2\pi}{a} b \tag{32}$$

In order to retain the previous single-wire analysis in its application to the grid surface, it will be necessary to space the grid wires so that the local electric field about any individual wire behaves as a $1/r$ field at least up to $r = R_a$, the effective capture radius. The circular sine and the hyperbolic sine functions are equal to each other for small arguments and can be approximated by the value of the argument. Thus, for $\theta = 0.3$, $\sin \theta = 0.2955$, and $\sinh \theta = 0.3045$.

The equipotentials given in equation (30) are then early circles for

$$\sin^2 \frac{\pi}{a} x + \sinh^2 \frac{\pi}{a} y \cong \left(\frac{\pi}{a}\right)^2 (x^2 + y^2) = \left(\frac{\pi}{a}\right)^2 r^2 \leq (0.3)^2; \qquad \frac{\pi}{a} r \leq 0.3 \tag{33}$$

When $r = R_a$, equation (33) will be made to hold and hence the grid spacing will be fixed in terms of the wire radius. The value of a_{min} from equation (34) is, therefore,

$$a_{min} = 10.5\ R_a \tag{34}$$

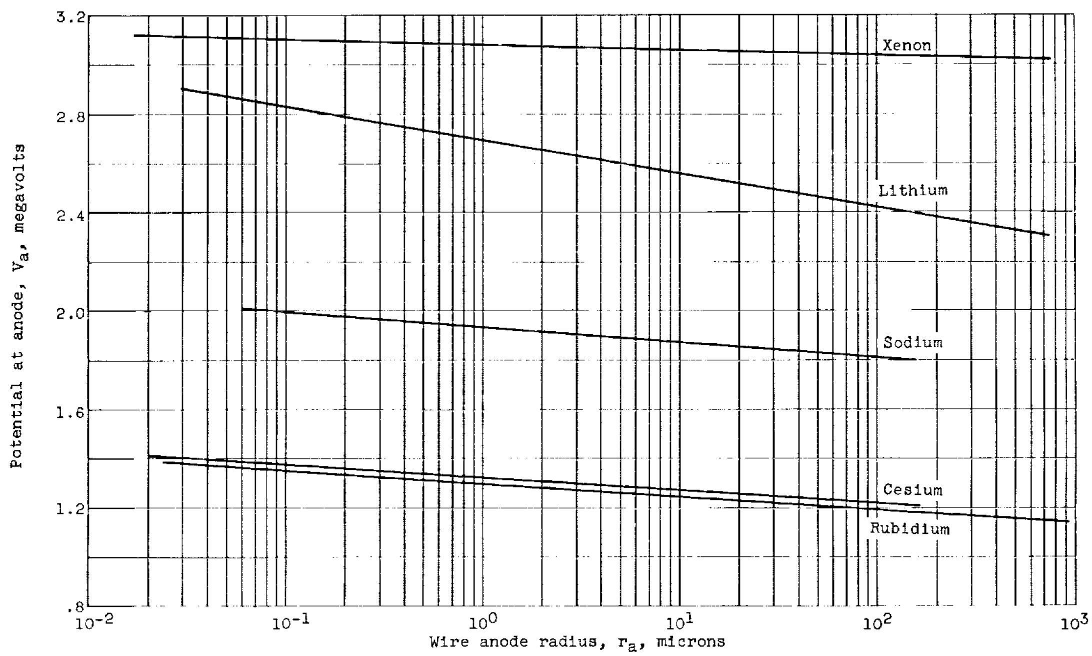


Figure 4. - Potential at anode as function of wire anode radius for 63 percent ionization efficiency.

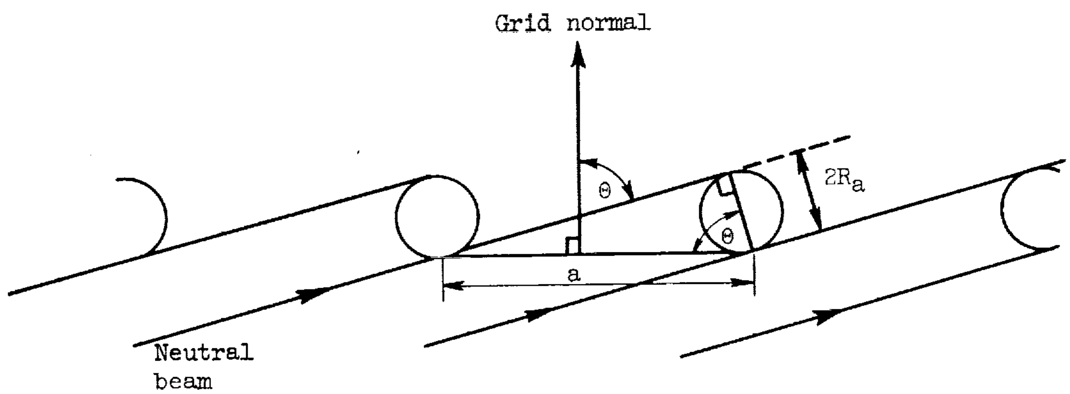
Figure 4 shows a plot of the grid voltage, given by equation (32), as a function of wire radius. The condition in equation (34), as well as the data given in figures 2 and 3, was used in calculating these potentials. The accelerator spacing b was chosen to be 2 millimeters, which is probably a practical minimum for the plane-diode engine.

Equation (34) is also used to yield the angle that a collimated beam of neutrals should make with the normal to the grid plane so that all the particles in the beam can pass within a capture area defined by R_a. The relation between the effective capture radius and the angle of incidence of the neutral beam

$$\theta \geq \cos^{-1} \frac{2R_a}{a} \cong 79^{\circ} \tag{35}$$

can be derived from sketch (c).

A problem peculiar to the ionization of a gas by field emission arises when it is desired to calculate the maximum current density attainable. For space-charged-limited currents the boundary condition is such that the field strength vanishes on the emitting surface. This condition can never be applied to the isolated wire or point emitter in field emission, for ionization quickly ceases when the field drops below a certain level. The situation is somewhat different, however, for a grid of wires. Equation (30) shows that the potential of the grid surface very rapidly approaches that of a plane. It is estimated in appendix C that the saturation of this plane diode results in only a 3.7-percent saturation of the individual grid wires. The local fields about the individual wires should hardly be affected.



(c)

CONCLUDING REMARKS

The voltages necessary for the successful operation of an engine of this design that employs the field-ion emission technique seem to be beyond present capabilities, at least for monatomic gaseous propellants. Besides the problem of attaining high voltages, a related problem presents itself, namely, the high specific impulse that results from these low-mass particles being accelerated to an excessive velocity. Theoretically this problem could be alleviated by the use of an accel-decel system. Heavier particles such as molecules or colloids would also give a lower specific impulse and would therefore be more adaptable to a field-emission engine though the former would introduce fragmentation problems in the high field. The greater polarizability and the lower thermal velocities of the colloids will result in somewhat larger cross sections for ionization. Perhaps further study will show that field emission is a workable means of charging these particles.

Despite the pessimistic results for the particular elements studied in this report, it is felt that field-emission ionization is still an attractive method of producing ions. The fact remains that ions can be produced without excessive difficulty by using needle-shaped anodes and only moderate voltages. The basic problem in electrostatic engines will be containment, that is, keeping the neutral particles in the vicinity of the high field for a long enough time.

Lewis Research Center
National Aeronautics and Space Administration
Cleveland, Ohio, September 27, 1962

APPENDIX A

SYMBOLS

A	atomic weight per ionic charge
a	potential turning point, grid spacing
b	potential turning point, accelerating distance
$C_{\pm}$	function defined in eqs. (B2) and (B3)
D	penetration probability
E	initial thermal energy of atom; complete elliptic integral of second kind
e	electronic charge
$\mathcal{E}$	ionization potential
$\mathcal{F}$	applied electric field
$\mathcal{F}_a$	electric field at cylindrical anode surface
$\hbar$	Planck's constant divided by 2π
J_c	space-charge-limited current per unit length for diverging cylindrical flow
J_N	flux density of neutral beam
J_p	plane-diode current per unit length of grid emitter
j	current density
K	a constant, complete elliptic integral of first kind
K_1,K_2,K_3	separation constants (see eq. (3))
k	Boltzmann constant
$L(\xi),H(\eta),\Phi(\varphi)$	functions defined in eqs. (3a), (3b), and (3c)
l	angular momentum and orbital quantum number
m_e	electronic mass
m_p	particle mass
$\dot{N}$	number of particle collisions with emitter wire per second

n	principal quantum number
P	probability of ionization
R_a	effective capture radius for ionization
r	spherical coordinate; range force
r_a	wire anode radius
s	impact parameter
T	temperature
t	time
U	potential energy
u	dummy variable
V	electrostatic potential
V_a	potential at anode
W	complex potential
Z	atomic number
x,y,z	Cartesian coordinates
y	function defined in eq. (19)
z	$x + iy$
α	polarizability
β	Langmuir function for cylinders
ϵ	field energy of dipole in units of kT
ϵ_0	dielectric constant of a vacuum
ϵ_1	field energy of dipole in units of kT at boiling point and $\mathscr{F} = 1v/A$
Θ	angle of incidence of neutral beam
θ	azimuthal angle
μ	reduced mass
ν	orbital frequency

ξ, η, φ parabolic coordinates

τ ionization lifetime

ψ wave function, solution of Schrödinger's equation

Subscripts:

max maximum

min minimum

APPENDIX B

SOLUTION OF TRANSMISSION PROBABILITY

According to the W.K.B. approximation, the lifetime of an atom is obtained by combining equations (5) and (6) and carrying out the integration:

$$\tau = \nu^{-1} \exp\left[\frac{2\sqrt{2m_e}}{\hbar} \int_a^b \left(e\mathcal{E} - e\mathcal{F}z - \frac{e^2}{4\pi\epsilon_0 z}\right)^{1/2} dz\right]$$

$$= \nu^{-1} \exp\left[\frac{2}{\hbar} \sqrt{2m_e e}\ \mathcal{F}^{1/2} \int_a^b \left(-z^2 + \frac{\mathcal{E}}{\mathcal{F}} z - \frac{e}{4\pi\epsilon_0 \mathcal{F}}\right)^{1/2} z^{-1/2}\, dz\right] \tag{B1}$$

The roots of the integrand are:

$$a = \frac{\mathcal{E}}{2\mathcal{F}}\left(1 - \sqrt{1 - \frac{e\mathcal{F}}{\pi\epsilon_0 \mathcal{E}^2}}\right) \equiv \frac{\mathcal{E}}{2\mathcal{F}} C_- \tag{B2}$$

$$b = \frac{\mathcal{E}}{2\mathcal{F}}\left(1 + \sqrt{1 - \frac{e\mathcal{F}}{\pi\epsilon_0 \mathcal{E}^2}}\right) \equiv \frac{\mathcal{E}}{2\mathcal{F}} C_+ \tag{B3}$$

With $z^{1/2} = y$ and equations (B2) and (B3) substituted, equation (B1) takes the form

$$\tau = \nu^{-1} \exp\left[\frac{4}{\hbar} \sqrt{2m_e e}\ \mathcal{F}^{1/2} \int_{\sqrt{a}}^{\sqrt{b}} \sqrt{(b - y^2)(y^2 - a)}\, dy\right] \tag{B4}$$

This integral can be written in terms of elliptic functions (ref. 6, p. 57, formula 218.11)

$$\int_{\sqrt{a}}^{\sqrt{b}} \sqrt{(b - y^2)(y^2 - a)}\, dy = (b - a)^2 \frac{1}{\sqrt{b}} \int_0^{u_1} \mathrm{sn}^2 u\ \mathrm{cn}^2 u\, du \tag{B5}$$

where sn and cn are the elliptic sine and cosine functions and $u_1 = K\left(\frac{\pi}{2}, k\right)$ is the complete elliptic integral of the first kind, in which

$$k = \sqrt{\frac{b - a}{b}} \tag{B6}$$

Integrating equation (B5) leaves

$$\frac{(b - a)^2}{\sqrt{b}} \frac{1}{3k^4} \Big[(2 - k^2)E(u) - 2(1 - k^2)u - k^2 \mathrm{sn}\, u\, \mathrm{cn}\, u\, \mathrm{dn}\, u\Big]_0^K$$

$$= \frac{1}{3} b^{1/2} \left[(a + b)E\left(\frac{\pi}{2}, k\right) - 2aK\left(\frac{\pi}{2}, k\right)\right] \qquad \text{(B7)}$$

In terms of $C_\pm$,

$$\frac{(b - a)^2}{\sqrt{b}} \frac{1}{3k^4} \Big[(2 - k^2)E(u) - 2(1 - k^2)u - k^2 \mathrm{sn}\, u\, \mathrm{cn}\, u\, \mathrm{dn}\, u\Big]_0^K$$

$$= \frac{1}{3\sqrt{2}} \frac{\mathcal{E}^{3/2}}{\mathcal{F}} C_+^{1/2} \left[E\left(\frac{\pi}{2}, k\right) - C_- K\left(\frac{\pi}{2}, k\right)\right] \qquad \text{(B8)}$$

and the final form of the lifetime is

$$\tau = \nu^{-1} \exp\left\{\frac{4\sqrt{m_e e}}{3\hbar} \frac{\mathcal{E}^{3/2}}{\mathcal{F}} C_+^{1/2} \left[E\left(\frac{\pi}{2}, k\right) - C_- K\left(\frac{\pi}{2}, k\right)\right]\right\} \qquad \text{(B9)}$$

APPENDIX C

SPACE-CHARGE-LIMITED CURRENT OF GRID EMITTER

It is desired to calculate the limiting current between the two cylinders r_a and R_a and find its fraction of the limiting plane-diode current per wire. The limited current density for a plane diode is given by Child's law (ref. 7):

$$j = 5.467\times10^{-8}A^{-1/2}V_a^{3/2}b^{-2}\ \text{amp/m}^2 \tag{C1}$$

The current per wire per unit length of grid is then

$$J_p = 2R_a j \tag{C2}$$

Substituting equation (C1) into (C2) yields

$$J_p = 1.093\times10^{-7}A^{-1/2}V_a^{3/2}b^{-2}R_a \tag{C3}$$

The space-charge-limited current per unit length for diverging cylindrical flow is given by the Langmuir relation (ref. 7):

$$J_c = 3.43\times10^{-7}A^{-1/2}(\Delta V)^{3/2}R_a^{-1}\beta^{-2} \tag{C4}$$

The ratio of these two currents measures the degree of current limitation of the grid wires in terms of a space-charge-limited diode:

$$\frac{J_p}{J_c} = 0.319\,\frac{V_a^{3/2}}{\Delta V^{3/2}}\,\frac{\beta^2R_a^2}{b^2} \tag{C5}$$

The term ΔV is the difference in potential between r_a and R_a, and

$$\Delta V = \mathscr{F}_a r_a \ln\frac{R_a}{r_a} \tag{C6}$$

The potential V_a is defined in equation (32). The worst case occurs when β and R_a are large and b is small; then J_p/J_c is at its largest.

With $b = 2\times10^{-3}$ meter, $r_a = 10^{-5}$ meter, $R_a/r_a = 10$, and $\beta \cong 1$, it is found that

$$\frac{J_p}{J_c} \cong 0.037 \tag{C7}$$

The amount of current needed to saturate the plane diode is at most only 3.7 percent of the current needed to saturate the individual grid wires.

REFERENCES

1. Weyl, Hermann: Über gewöhnliche Differentialgleichungen mit Singularitäten und die zugehörigen Entwicklungen willkürlicher Funktionen. Math. Ann., bd. 68, 1910, pp. 220-269.

2. Oppenheimer, J. R.: Three Notes on the Quantum Theory of Aperiodic Effects. Phys. Rev., vol. 31, Jan. 1928, pp. 66-81.

3. Frank, N. H., and Young, L. A.: Transmission of Electrons Through Potential Barriers. Phys. Rev., vol. 38, July 1931, pp. 80-85.

4. Bohm, D.: Quantum Theory. Prentice-Hall, 1951, p. 264.

5. Morse, P. M., and Feshback, H.: Methods of Theoretical Physics. Vol. II. McGraw-Hill Book Co., Inc., 1953, p. 1235.

6. Byrd, P. F., and Friedman, M. D.: Handbook of Eliptic Integrals for Engineers and Physicists. Springer-Veerlag (Berlin), 1954, p. 57.

7. Lockwood, David L., Mickelsen, William, and Hamza, Vladimir: Analytic Space-Charge Flow and Theoretical Electrostatic Rocket Engine Performance. Paper 2400-62, Am. Rocket Soc., Inc., 1962.

8. Landolt-Börnstein: Atom-und Molekularphysik, Teil 1, Springer-Veerlag (Berlin), 1950, p. 401.